4 Differentiation

4.1 The Slope of a Function

If a tax system is described (see **Section 1.1**) by the linear function

$$Y = 300 + 0.7X$$

then when X goes up by $£1$, Y goes up by $70p$, and tax, which is given by the function

$$T = 0.3X - 300$$

goes up by $30p$. In fact, for **any** increase in X, there is an increase in Y of 70% of the increase in X, and an increase in T of 30% of the increase in X. Formally, if X increases by an amount $\triangle X$ then Y increase from $Y = 300 + 0.7X$ to

$$\begin{aligned} Y + \Delta Y &= 300 + 0.7(X + \Delta X) \\ &= 300 + 0.7X + 0.7\Delta X = Y + 0.7\Delta X \end{aligned}$$

so that $\frac{\Delta Y}{\triangle X} = 0.7$. In a graph, this quantity is easily seen to measure the <u>slope</u> of the line.

For the general linear function

$$\begin{aligned} Y &= a + bX \\ \frac{\Delta Y}{\Delta X} &= b \end{aligned}$$

which means that the slope of the graph is the same at all values of X (which is why it is a straight line).

The same cannot be true of non-linear functions. For instance, if a ?rm's total costs are a function of the quantity of the good it produces, such that

$$C = Q^2$$

then the effects of costs of each unit increase in quantity are given by the table:

Q	0		1		2		3		4		5
C	0		1		4		9		16		25
ΔC		1		3		5		7		9	
ΔQ		1		1		1		1		1	

so that $\frac{\Delta C}{AQ}$ depends on the value of Q. But it also depends on the value of ΔQ, for when quantity goes from Q to $Q + \Delta Q$, cost goes from $C = Q^2$ to

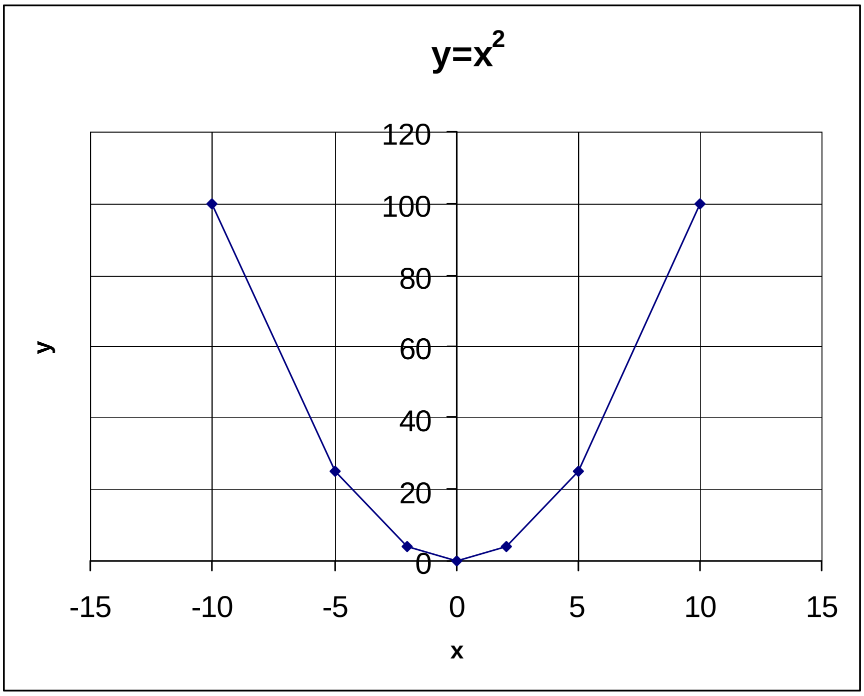
$$\begin{aligned} C + \Delta C &= (Q + \Delta Q)^2 = Q^2 + 2Q.\Delta Q + \Delta Q^2 \\ \text{so} \qquad \Delta C &= 2Q.\Delta Q + \Delta Q^2 \\ \text{and} \qquad \frac{\Delta C}{\Delta Q} &= 2Q + \Delta Q \end{aligned}$$

The term $\Delta C/\Delta Q$ can be regarded as the marginal cost of production, but the fact that it depends on how large is the increment ΔQ in output makes this de?nition unsatisfactory. To resolve this problem we let the output change become arbitrarily small, or, in other words, let ΔQ tend to zero (usually written as$\Delta Q \to O$). As this occurs $\Delta C/\Delta Q \to 2Q$, and we get a unique value for marginal cost at each level of output. For example, marginal cost would be 4 when $Q = 2$. In graphical terms this would correspond to the slope of the line (the tangent) that just touches the graph of $C = Q^2$at the point $Q = 2$.

Graph of $y = x^2$ and its slope.

Graph of: $y=x^2$

x	y
-10	100
-5	25
-2	4
0	0
2	4
5	25
10	100



4.2 The De?nition of a Derivative

The slope of a function at a point is otherwise known as the <u>derivative</u> of the function at this point, and the process of ?nding the derivative is called <u>differentiation</u>. In the above cost example we considered the ratio

$$\frac{\Delta C}{\Delta Q} = \frac{(Q+\Delta Q)^2 - Q^2}{\Delta Q} = 2Q + \Delta Q$$

and argues that $\Delta C/\Delta Q$ "tends in the limit" to $2Q$ as ΔQ tends to zero. This can be written formally as

$$\frac{dC}{dQ} = \underset{\Delta Q \to 0}{Limit} \frac{\Delta C}{\Delta Q} = 2Q$$

where $\frac{dC}{dQ}$ is the notation for the derivative of cost C with respect to output Q. All **marginal** concepts in economics (marginal cost, revenue etc.) are normally de?ned as **derivatives of the appropriate functions**.

The concept of a derivative can be applied to any function. The notation

$$Y = f(X)$$

is used to express the fact that Y is a function of X, without explicitly stating what kind of function (linear,quadratic,...) it is. Then the derivative is de?ned as

$$\frac{\Delta Y}{\Delta X} = \underset{\Delta X \to 0}{Limit} \frac{\Delta Y}{\Delta X} = \underset{\Delta X \to 0}{Limit} \left\{ \frac{f(X+\Delta X) - f(X)}{\Delta X} \right\}$$

Alternative notations include replacing ΔX by h, or writing $\frac{dY}{dX}$ as $\frac{df(X)}{dX}$, $f'(X), \frac{df}{dx}, f'(x), D(f(x)), y', Dy$.

Normally, functions are not given in a simple form, there are total cost functions that could take complicated forms as: $TC = 2Q^3 - 18Q^2 + 60Q + 50$. Terms of Q are multiplied by constants, have exponents, etc. Therefore, we will need a set of rules that helps us deal with such functions.

4.3 Rules for Differentiation

We have already seen that

$$\begin{aligned} Y &= x^2 \quad \text{implies} \quad \frac{dY}{dX} = 2X \\ Y &= aX + b \quad \text{implies} \quad \frac{dY}{dX} = a \end{aligned}$$

from which it also follows that $\frac{dY}{dX} = O$ when $Y = c$, a constant. Derivatives of more complicated functions can be obtained by using two simple rules of differentiation which follow directly from the above de?nition of derivatives.

Rule 1:

$$\begin{aligned} \text{If} \quad Y &= f_1(X) + f_2(X), \\ \frac{dY}{dX} &= \frac{df_1(X)}{dx} + \frac{df_2(X)}{dx} \end{aligned}$$

i.e., the derivative of the sum of two functions is the sum of the derivatives of the functions.

Rule 2:

$$\begin{aligned} \text{If} \quad Y &= af(X) \quad \text{where } a \text{ is a constant,} \\ \frac{dY}{dX} &= a\frac{df(X)}{dx} \end{aligned}$$

These two rules, together with what we already know about derivatives, imply that

$$\text{If} \quad Y = ax^2 + bX + c \quad \text{then} \quad \frac{dY}{dX} = 2ax + b.$$

Further rules of differentiation allow us to widen considerably the range of functions that can be differentiated. Proofs of the rules will not be given: interested students can ?nd them in textbooks.

Rule 3 (the product rule): If $Y = U.V.$ where U and V are functions of X, then Y is a function of X and

$$\frac{dY}{dX} = U\frac{dV}{dX} + V\frac{dU}{dX}$$

Examples:

1. If $Y = (X+2) \cdot (aX^2 + bX)$, $\frac{dY}{dX} = (X+2) \cdot (2aX + b) + (aX^2 + bX)$
2. If $Y = X^3 = X^2 \cdot X$, $\frac{dY}{dX} = X^2 + X \cdot 2X = 3X^2$

Rule 4: (the quotient rule): If $Y = \frac{U}{V}$ where U and V are functions of X, then Y is a function of X and

$$\frac{dY}{dX} = \frac{V\frac{dU}{dx} - U\frac{dV}{DX}}{V^2}$$

Examples:

1. If $Y = \frac{(X+2)}{(X+4)}$, $\frac{dY}{dX} = \frac{2}{(X+4)^2}$
2. If $Y = X^{-2}$, $\frac{dY}{dX} = -2X^{-3}$

Rule 5: If $Y = X^n$ where n is any number

$$\frac{dY}{dX} = nX^{n-1}$$

Example: The functions $X^2, X, X^\circ = 1, X^3, X^{-2}$ have already been shown by different methods to satisfy this rule.

$$\text{If } Y = X^{\frac{1}{3}}, \frac{dY}{dX} = \frac{1}{3}X^{-\frac{2}{3}}$$

Rule 6: If Y is a function of X in such a way that X is also a function of Y then

$$\frac{dX}{dY} = \frac{1}{\frac{dY}{dX}} = \left(\frac{dY}{dX}\right)^{-1}$$

Example: If $Y = x^{\frac{1}{3}}$,then $X = Y^3$. So $\frac{dX}{dY} = 3Y^2$and $\frac{dY}{dX} = \left(\frac{dX}{dY}\right)^{-1} = \frac{1}{3Y^2} = \frac{1}{3}X^{-2/3}$.

This provides another way of obtaining the result indicated in the previous example.

Rule 7: If Y is a function of V and V is a function of X, then Y is a function of X and

$$\frac{dY}{dX} = \frac{dY}{dV}.\frac{dV}{dX}$$

Example: If $Y = (ax^2 + bX)^{\frac{1}{2}}, \frac{dY}{dX} = \frac{1}{2}(aX^2 + bX)^{-\frac{1}{2}}(2aX + b)$

These rules of differentiation are sometimes useful even when we do not know what kind of function we are dealing with. For example, if all we know is that the price of a ?rm's product is a function of the quantity it sells, i.e., the demand curve is $P = f(Q)$, we can derive the marginal revenue. Total revenue is

$$R = Q.P = Q.f(Q)$$

and Rule 3 implies that marginal revenue is

$$\frac{dR}{dQ} = Q\frac{df(Q)}{dQ} + f(Q)$$

Higher derivatives:

Similarly, the second derivative is obtained by differentiating the ?rst derivative again with respect to x :

$f''(x) = \frac{d}{dx}(f'(x))$ i.e., if $f(x) = 3x^2 - 5x + 4$, then, $f'(x) = 6x - 5$, and $f''(x) = 6$.

And, the third derivative would be obtained by differentiating the second derivative with respect to x and so on and so forth.

Examples:

1. Simplify each of the following so that the differentiation may be carried out using the power rule, and obtain ?rst and

second derivatives: a) $\frac{Q^{10}}{Q^2}$,b) $4\frac{\sqrt{Q^{10}}}{Q^2}$,c) $\frac{x^2+x}{x^3}$, d) $\sqrt{5^2Q^3}$, e) $3\left(\frac{Q^2+2Q}{Q}\right)+2$

a) $\frac{Q^{10}}{Q^2}=Q^{10-2}=Q^8 \quad f'(Q)=8Q^7 \quad f''(Q)=56Q^6$

b) $4\frac{\sqrt{Q^{10}}}{Q^2}=4Q^{\frac{10}{2}-2}=4Q^{5-2}=4Q^3 \quad f'(Q)=12Q^2 \quad f''(Q)=24Q$

c) $\frac{x^2+x}{x^3}=x^{2-3}+x^{1-3}=x^{-1}+x^{-2} \quad f'(Q)=-1x^{-1-1}-2x^{-2-1}=-x^{-2}-2x^{-3}$
$f''(Q)=2x+6x$

d) $\sqrt{5^2Q^3}=5Q^{\frac{3}{2}} \quad f'(Q)=5\frac{3}{2}Q^{\frac{3}{2}-1}=\frac{15}{2}Q^{\frac{1}{2}} \quad f''(Q)=\frac{15}{2}\frac{1}{2}Q^{\frac{1}{2}-1}=\frac{15}{4}Q^{-\frac{1}{2}}$

e) $3\left(\frac{Q^2+2Q}{Q}\right)+2=3\frac{Q^2}{Q}+3\frac{2Q}{Q}+2=3Q^{2-1}+6Q^{1-1}+2=3Q^1+6+2=3Q+8$
$f'(Q)=3 \quad f''(Q)=0$

2. Find the ?rst, second and third derivatives of the following total cost function: $TC=\frac{Q^3}{5}-8Q^2+\frac{5Q}{2}+180$
$TC'(Q)=\frac{dTC(Q)}{dQ}=\frac{3}{5}Q^{3-1}-16Q^{2-1}+\frac{5}{2}Q^{1-1}=\frac{3}{5}Q^2-16Q^1+\frac{5}{2}Q^0=\frac{3}{5}Q^2-16Q+\frac{5}{2}$

$TC''(Q)=\frac{d}{dQ}(\frac{dTC(Q)}{dQ})=\frac{6}{5}Q^{2-1}-16Q^{1-1}=\frac{6}{5}Q^1-16$

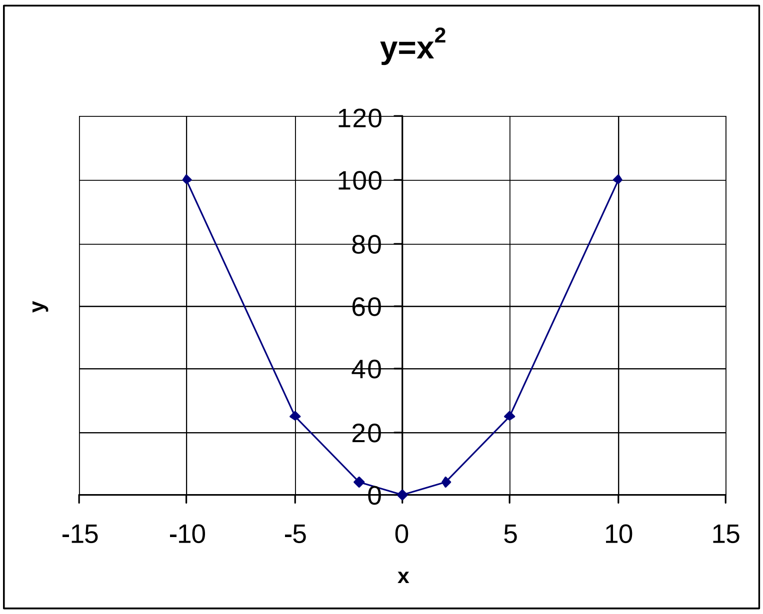
4.4 Use of differentiation to determine whether the slope of a function is increasing or decreasing:

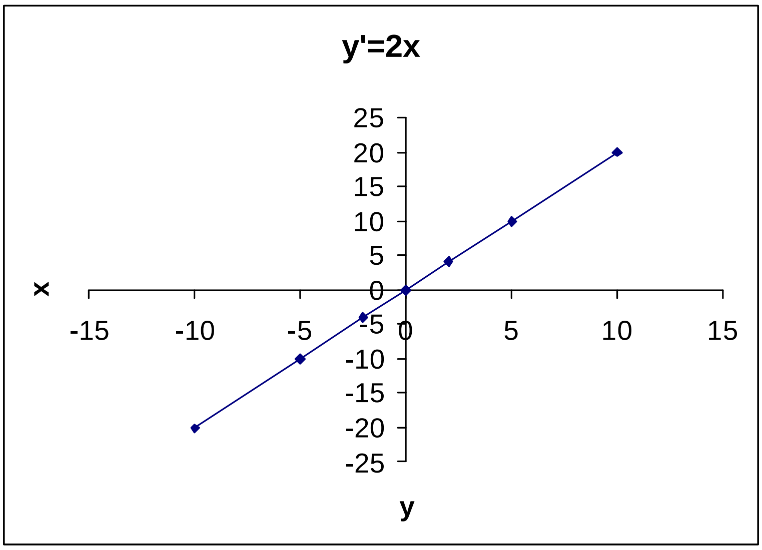
The sign of the derivative of y, $\frac{dy}{dx}$, indicates whether y is increasing or decreasing as x is increasing because:

when y is increasing, for $x_1 < x_2$: $f(x_1) < f(x_2)$, then, $\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{+}{+}>o \Rightarrow y'>0$ (the slope is positive and so is the ?rst derivative)

when y is decreasing, for $x_1 < x_2$: $f(x_1) > f(x_2)$, then, $\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{-}{+}<o \Rightarrow y \Rightarrow y'<0$ (the slope is negative and so is the ?rst derivative)

Think of the example $y = x^2$





Think of example in Addendum (Sketching the graph of the derivative).

4.5

Elasticity

If demand is a function of income, the **income elasticity of demand** is de?ned as

$$e_Y = \frac{\text{proportional change in demand}}{\text{proportional change in income}} = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta Y}{Y}} = \frac{Y}{Q}\frac{\Delta Q}{\Delta Y}$$

This will vary in general with ΔY (in the same way as marginal cost previously varied with ΔQ). A more precise de?nition that does not depend on the increment ΔY is therefore:

$$e_Y = \frac{Y}{Q}\frac{dQ}{dY}$$

Example: The function $Q = aY^b$ has income elasticity constant and equal to b.

Proof: $\frac{dQ}{dY} = abY^{b-1}$. Then, $e_Y = \frac{Y}{Q}\frac{dQ}{dY} = \frac{Y}{Q}abY^{b-1} = \frac{b(aY^{b-1+1})}{aY^b} = b$

The **price elasticity** of demand is de?ned in a similar way

$$e_d = \frac{\text{proportional change in demand}}{\text{proportional change in price}} = \frac{P}{Q}\frac{dQ}{dP} \approx \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}}$$

Example: If the demand curve is given by $Q = aP^{-b}$,then the elasticity of demand is $-b$ at all price levels.

Proof: $\frac{dQ}{dP} = -abP^{-b-1}$. Then, $e_Y = \frac{P}{Q}\frac{dQ}{dP} = -\frac{P}{Q}abP^{-b-1} = -\frac{b(aP^{-b-1+1})}{aP^{-b}} = b$

The value of e_d is expected to be negative (downward sloping demand curve), but statements involving elasticities usually disregard the sign and refer only to the magnitude of the number ("more elastic" therefore indicates a larger sized negative number for e_d). The magnitude of a number of X, without regard to its sign, is denoted mathematically by $|X|$ (the absolute value of X)

The following results about price elasticity of demand are now fairly straight-forward to prove.

1. If $|e_d| > 1$, a price rise reduces revenue. If $|e_d| < 1$, a price rise raises revenue. If $e_d = 1$, revenue is constant.
2. If a producer chooses a price at which marginal cost equals marginal revenue and if marginal cost is positive, $|e_d| > 1$ at the chosen price.
3. The price set by a producer who equates marginal cost and marginal revenue will be

$$P = MC\frac{e_d}{1 + e_d}$$

Proof:

1. $|e_d| > 1 \Rightarrow |e_d| \approx \left|\frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}}\right| > 1 => \left|\frac{\%\Delta Q}{\%\Delta P}\right| > 1 \Rightarrow |\%\Delta Q| > \%\Delta P$. Thus, an increase in price by 1%, produces a fall in the quantity sold bigger than 1%! Thus, total revenue will suffer.

   By the same reasoning, if $|e_d| < 1$, $\left|\frac{\%\Delta Q}{\%\Delta P}\right| < 1$, and since increasing prices a % will provoke a fall in quantity smaller than that %, total revenue will increase.

If $|e_d| = 1$, an increase in prices is perfectly compensated by a decrease in quantity sold, so that revenues do not change.

2. If a producer chooses a price at which marginal cost equals marginal revenue and if marginal cost is positive, $|e_d| > 1$ at the chosen price.

   $MC = MR \Rightarrow \frac{dC}{dQ} = \frac{d}{dQ}(PQ) = \frac{dP}{dQ}Q + P => \frac{dC}{dQ}\frac{1}{P} = \frac{dP}{dQ}\frac{Q}{P} + 1 = \frac{1}{e_d} + 1$

   $\frac{1}{e_d} = \frac{MC}{P} - 1 => |e_d| = \frac{P}{P-MC} > 1$

3. From before, $MC = MR \Rightarrow MC = P\left(\frac{1}{e_d} + 1\right) \Rightarrow P = MC(\frac{e_d}{e_d+1})$

Examples (from Bradley and Patton):

1. When the price of a good is £20, the price elasticity of demand is -0.7. Calculate the percentage change in demand Q when :

   a) The price increases by 5%.

   b) The price increases by 8%.

   a) $e_d = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} \Rightarrow -0.7 = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}}$

   If $\frac{\Delta P}{P} = 5\%$, it means that $\frac{\Delta P}{20} = 0.05$. So, $-0.7 = \frac{\frac{\Delta Q}{Q}}{0.05}$. Thus, $\frac{\Delta Q}{Q} = (-0.7) \times 0.05 = -0.035$. In other words, the percentage change in demand is -3.5%.

   b) $e_d = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} \Rightarrow -0.7 = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} = \frac{\frac{\Delta Q}{Q}}{0.08}$. So, $\frac{\Delta Q}{Q} = (-0.7) \times 0.08 = -0.056$. In other words, the percentage change in demand is -5.6%.

2. For the demand functions (i) $P = a - bQ$ (ii) $Q = c - dP$

   a) Derive the expressions for the price elasticity of demand in terms of Q only.

   b) Calculate e_d for the following demand function at $Q = 100; 500; 900$:

   $P = 60 - 0.5Q$

   a) First, we express the demand function as a function of P :

   $P = a - bQ \Rightarrow bQ = a - P \Rightarrow Q = \frac{a}{b} - \frac{1}{b}P$

   Second, we calculate $\frac{dQ}{dP} = -\frac{1}{b}$.

   Last, we know that : $e_d = \frac{dQ}{dP}\frac{P}{Q} = -\frac{1}{b}\frac{P}{Q} = -\frac{1}{b}\frac{a-bQ}{Q} = 1 - \frac{a}{b}\frac{1}{Q}$

   For $Q = c - dP$, $\frac{dQ}{dP} = -d$. Then, $e_d = \frac{dQ}{dP}\frac{P}{Q} = -d\frac{P}{Q} = -d\frac{\frac{c}{d} - \frac{Q}{d}}{Q} = -\frac{d \times c}{d \times Q} + \frac{dQ}{dQ} = 1 - \frac{c}{Q}$

3. $P = 90 - 0.05Q$ is the demand function for graphics calculators in an engineering college.

   a) Derive expressions for e_d in terms of 1) P only, 2) Q only.

   b) Calculate the value of e_d when the calculators are priced at $P = £20; £30; £70$.

c) Determine the number of calculators demanded when $e = -1; e_d = 0$.

a) $P = 90 - 0.05Q$, then $\frac{dP}{dQ} = -0.05$, and also $Q = \frac{90}{0.05} - \frac{P}{0.05} = 1800 - 20P$

$e_d = \frac{dQ}{dP}\frac{P}{Q} = \frac{1}{\frac{dP}{dQ}}\frac{P}{Q} = -\frac{1}{0.05}\frac{P}{Q} = -\frac{1}{0.05}\frac{P}{1800-20P} = -\frac{P}{90-P} = \frac{P}{P-90}$

$e_d = \frac{dQ}{dP}\frac{P}{Q} = -\frac{1}{0.05}\frac{90-0.05Q}{Q} = -\frac{90}{0.05Q} + \frac{0.05Q}{0.05Q} = 1 - \frac{1800}{Q}$

b) $e_d(P = 20) = \frac{P}{P-90} = \frac{20}{20-90} = -0.28$

$e_d(P = 30) = \frac{P}{P-90} = \frac{30}{30-90} = -0.50$

$e_d(P = 70) = \frac{P}{P-90} = \frac{70}{70-90} = -3.5$

c) If $e_d = -1$, it means that $e_d = 1 - \frac{1800}{Q} = -1$. Thus, $\frac{1800}{Q} = 2$, and $Q = 900$.

If $e_d = -0$, it means that $e_d = 1 - \frac{1800}{Q} = 0$. Thus, $\frac{1800}{Q} = 1$, and $Q = 1800$.